Convexity of the Moore–Penrose inverse

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Abstract

In this talk I shall review convexity properties of the Moore–Penrose inverse matrix function. Convexity refers here to the partial (Löwner) ordering of Hermitian matrices. The convexity result for positive definite matrices was no doubt known to Karl Löwner and his students in the 1930s (and perhaps more broadly), but appeared in print later due to its use in problems of optimum experimental design and multivariate probability (Chebyshev-type) inequalities. Motivated in part by such optimality problems, various extensions to nonnegative definite matrices using generalized inverses have also been derived; see Section 1 of [1] for a survey of such convexity results for positive definite as well as nonnegative definite matrices. Also, in [1] the question was raised whether in the convexity inequality (with $\overline{\lambda} := 1 - \lambda$)

$$(\lambda A + \overline{\lambda} B)^{+} \leq \lambda A^{+} + \overline{\lambda} B^{+}$$

the inequality holding for a single $\lambda \in]0, 1[$ is enough to guarantee its validity for all $\lambda \in]0, 1[$. In [2] this was shown to be the case, the proof relying on a general result on "t–convexity" based on Rode's abstract version of the classical (Helly–)Hahn–Banach Theorem. Time permitting I will present alternative simpler proofs which avoid the use of this general result.

Keywords

Jensen convexity, midpoint convexity, rational convexity.

References

[1] Nordström, K. (2011). Convexity of the inverse and Moore–Penrose inverse. *Linear Algebra Appl. 434*, 1489–1512.

[2] Nordström, K. (2018). A note on the convexity of the Moore–Penrose inverse. *Linear Algebra Appl. 538*, 143–148.